Gedanken 2005 (Finals)

Showcase 2005
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(Marks given in brackets indicate maximum marks for correctness and elegance respectively. In questions where an answer has been asked for, proof has to be provided for getting any credit.)

Hope you enjoyed the first day of this interesting conference. Now lets move on to the second day's proceedings. Second day's discussion is started by another old man in the group. The great mathematician Pythagoras.

1 Pythagoras and his irrationals

The ageing Pythagoras said

> You must be knowing that the discovery of irrational numbers is attributed to the Pythagorean *Hippasus of Metapontum*. But it was only yesterday that I came to know of an amazing fact about them. A mathematician friend told me that he had been able to prove that there are irrational numbers a and b such that a^b is rational. He actually used a very interesting number, $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}$ to prove this fact!

Can *you* prove what Pythagoras' friend had proved?

(5+9)

2 Turing's sorting problem

Now it was the turn of the computer scientists to hog the limelight. And better to talk about computer science than Alan Turing! So Turing decided to give a problem to the mathematicians- and it was as simple as sorting! He said

> I will give you an array (sequence) of n numbers to sort. However, the only operation you are allowed to do is to flip (reverse the order of) a prefix of the array (subsequence of consecutive numbers starting from the first number in the current array). Each such flip shall incur a cost of 1 unit. To make things simple for you, I have decided to tell you the maximum of any subset of the numbers, if you need it, free of cost.

As an example, consider the array {2, 3, 1, 4}. Then flip the first 2 elements to get {3, 2, 1, 4}, Now flip the first 3 elements to get {1, 2, 3, 4}, which is the sorted array. So, for this case, the cost of sorting is 2 flips. Give an algorithm which performs the above task in number of flips which is linear (or better!) in n (the size of the given array of numbers) for any arbitrary sequence of input numbers.

(5+5)

3 Fermat's generalized divisibility test

Fermat, who had kept silent till now, finally joined the discussion.

> I have been thinking about a rather interesting problem all this while! All of you must have seen the divisibility test for 7 at some stage in life. It basically involves doubling the least significant (rightmost) digit of the decimal representation of the number whose divisibility is being checked and subtracting the product from the remaining digits. It is claimed that the new number is divisible by 7 if the original number was divisible by 7 and vice-versa. For example, to check divisibility of 1127 by 7, convert to 112 - 2*7 = 98, which is further converted to 9 - 8*2 = -7, which is divisible by 7; hence, 1127 is divisible by 7. Now, I am thinking of generalizing this strategy- more precisely, given two positive integers a and b, and the fact that any number n is divisible by a if and only if the number formed by subtracting b times the least significant digit of n from the remaining digits of n is also divisible by a, can we specify some relation between a and b?

Can you help Fermat?

(10+0)

4 Polya's students

George Polya was as renowned a researcher as a teacher. So when he decided to present a problem, everyone knew it had to come straight out of the classroom. Polya began

> I gave a test to my students this weekend where a very curious thing happened. I had given k questions in increasing order of difficulty. Now, it so happened that anyone who solved question j also solved questions *1* to *j-1*, which were easier than question j; and curiously enough, this rule held for all j where $1 \leq j \leq k$ and for all students. Now, if n students took the test, can you find the number of ways the students could have answered the test without violating the above rule?

Note that students are distinct and two answering patterns shall be considered different if even one student answers a different set of questions. It is possible that a student does not answer any question at all.
Can you find the number of possible ways in which Polya's students can answer the questions?

(2+8)

5 Riemann's integrals

Riemann had been quiet all this time. When he did say something, it was obvious that it had to do with his favourite integrals!

> I have been thinking about a rather interesting problem for the last few days. And it was only yesterday that I tumbled upon a very elegant solution to it! So, I thought about giving the problem to you as well. Can you find the value of
> $$\int_0^1 \int_0^1 \dots \int_0^1 [\, x_1 + x_2 + \dots + x_n \,]\, dx_1\, dx_2 \dots dx_n,$$
> where $[x]$ denotes the greatest integer function of x (greatest integer $\leq x$).

Can you find the value of the integral?

(5+5)

6 Euler and his curious numbers

Euler, the Swiss mathematician of considerable repute came up with an interesting problem.

> Prove that the number
> $$[(1+\sqrt{3})^{2n+1}]$$
> is divisible by 2^{n+1}, where $[x]$ denotes the greatest integer function of x (greatest integer $\leq x$).

Can you solve Euler's problem?

(5+3)

7 Einstein the physicist

Einstein was famous for his *gedankens*. But when he rose to present a problem, it appeared to be of a very practical nature.

> A few days ago an experimental physicist friend of mine had given 4 identical instruments which he had constructed. He wanted to find the maximum integral voltage that the instruments could survive. All he was able to say was that the maximum voltage the instruments could survive was between $1V$ and $220V$. The problem was that he had lost his design and therefore could not provide any more replicas of his instruments. He needs 1 instrument to use in the future, but he allowed me to burn out the other 3 instruments to find the rating. I performed a series of experiments subjecting the instruments to different voltages. Can you tell me how I found the rating using the minimum number of experiments?

Give a solution to Einstein's problem using the minimum number of experiments.

(10+0)

8 Newton the parliamentarian

It was Newton's turn now. He said

> As all of you probably know, I have recently become a parliamentarian. And I have found out that politics is a rather nasty business! In fact, I have been astonished to know that some of our parliamentarians are refusing to share the same floor of the house with some of their bitter rivals. Fortunately enough, each member has at most 3 rivals who he is not ready to sit with in the same house. Now, as you know, we have only two houses of parliament, the House of Lords and the House of Commons. So, the leaders of the two houses met me yesterday and asked me whether we would be able to form the parliament at all assuming that any member can be placed in any house. They also told me that all the members have reluctantly agreed to share the same house with atmost 1 of their rivals each. It immediately became clear to me that there shouldn't be a problem in assigning members to the two houses! Can you see why?

Can you see why there shouldn't be a problem in forming the British parliament?

(2+4)

9 Aryabhatta's space odyssey

Aryabhatta, the renowned astronomer, was as usual, ahead of his times. So his problem looked into the future!

> This is problem from 2020 AD. You have 3000 gallons of space-craft fuel and a newly designed fuel-efficient spacecraft which consumes just 1 gallon of fuel in moving 1 lightyear. You want to transport this fuel using this spacecraft from earth to the star, *Alpha Centauri*, which is 1000 lightyears away. The only bad news is that the spacecraft has a capacity of only 1000 gallons. Give a strategy to transport the maximum fuel from earth to *Alpha Centauri*, given that there are space-stations all along the way from earth to *Alpha Centauri*, spaced 1 light year apart.

Assuming that the space stations do not have any fuel initially but have provision for storing any amount of fuel and (obviously) do not consume any fuel, solve Aryabhatta's riddle.

(10+0)

10 Gauss has the final say

Finally, after all the great minds had spoken, the conference came to an end. And who better to deliver the closing address than the greatest of them all, Carl Friedrich Gauss! So he rose

> Dear Friends,
> Over the last two days, we have had the most enlightening of discussions in almost all areas that we are passionate about. So, as I bring down the curtains on this great event, let me give you something to work on as we go on our own ways. Can you prove that there exists a matrix with integral elements such that its determinant is equal to 1 *if and only if* $gcd(a_1, a_2, ..., a_n) = 1$, where $a_1, a_2, ..., a_n$ are the elements of the first row of the matrix. Thanking all of you for your co-operation and active participation, I declare the conference closed. Thank you.

As we thank you for your participation in Gedanken 2005, can you call upon your intellectual faculties for a last time and solve Gauss' problem?

(6+6)

(Disclaimer: The above conversation is entirely hypothetical and may be historically inaccurate.)